Minimum Weight Sandwich Beam Design

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Theme

THE increasing demand for lightweight, highly reliable structures is forcing structural designers into carefully examining nontraditional design techniques and materials. The high-strength/low-density characteristics of sandwich materials offer great potential for modern structural design. In the present context, sandwich elements are defined as structural members comprising a core, of low density and relatively low tensile strength, bonded between two relatively thin faces made of high density, high tensile strength material.

Up to the present time, most applications of sandwich elements have utilized constant core and face thickness designs. As fabrication techniques improve and the demand for low weight intensifies, it is likely that sandwich elements with non-constant cross sections will be utilized. It is the purpose of this Synoptic to present a method which can be used to determine the minimum weight design of sandwich beams with rectangular cross section and constant width. The variable core thickness and face thicknesses are the control (or design) variables.

Since in most design situations, the final design is subjected to a variety of constraints, the present formulation includes the possibility of inequality constraints on maximum stress, and minimum and maximum core and face thicknesses. In addition, the present formulation includes the constraint that the total deflection at some specified point must equal some specified value.

Contents

Some problems in the minimum weight design of sandwich beams have been considered in Prager[1] and others. (See Ref. 2 for a complete bibliography.) In this series of papers, an optimality rule was developed to provide necessary conditions for minimum weight beams subjected to specified loads and specified maximum deflection. No stress limitations were imposed. Hahn, Citron, and Alspaugh[3] treated minimum weight design of elastic and plastic statically determinate and statically indeterminate homogeneous beams and frames subject to constraints on maximum bending stress, maximum shearing stress, and maximum deflection. The width of the beam or frame was used as the control variable.

In this work, the beam is assumed to be of rectangular cross section of unit width comprising two thin faces of equal thickness f separated by a low density core of thickness c. Both the face and core materials are isotropic. The face material is of high strength and high weight density ρ_1. The core material is of low strength and lower weight density ρ_2. The face is very thin compared with the core, i.e., $f/c \ll 1$. The faces resist the longitudinal loads and the core resists the transverse load. It is assumed that the shear stress in the core is constant across its depth.

The governing equations for sandwich beams can be written in terms of partial deflections w_b and w_s. w_b is the deflection of the sandwich beam due to bending only, it occurs if G_c, the shear modulus of the core, is infinite. w_s is the deflection of the sandwich beam due to shear deformation of its core, it occurs if E_f, the Young's modulus of the face, is infinite. Huang[2] presents a detailed development of partial deflections.

The total deflection w is the sum of partial deflections w_b and w_s.

$$w = w_b + w_s \tag{1}$$

The governing equations for sandwich beams can be written in terms of partial deflection as

$$d^2 w_b/dx^2 = M(x)/B \tag{2}$$

and

$$dw_s/dx = Q(x)/S \tag{3}$$

where $M(x)$ is the bending moment and $Q(x)$ is the shearing force. The bending stress across the face at a point x is assumed to be uniform. Thus

$$\sigma(x) = M(x)/f(f+c) \tag{4}$$

The shear stress at a point x is also assumed to be uniform across the core thickness

$$\tau(x) = Q(x)/(f+c) \tag{5}$$

The boundary conditions for a free edge are $M = 0$ and $Q = 0$, for a simply supported edge $w = 0$ and $M = 0$, and for a clamped edge $w = 0$ and $\dot{w} - \gamma = 0$, where γ is the average shear strain.

In terms of partial deflections w_b and w_s, the boundary conditions are

$$\text{free edge} \quad \ddot{w}_b = 0 \quad \text{and} \quad \dot{w}_s = 0 \tag{6}$$

$$\text{simply supported edge} \quad w = w_b + w_s = 0 \quad \text{and} \quad \ddot{w}_b = 0 \tag{7}$$

$$\text{clamped edge} \quad w = w_b + w_s = 0 \quad \text{and} \quad \dot{w}_b = 0 \tag{8}$$

in Eq. (8), it has been assumed that the local bending stiffnesses of faces are negligible.

For a sandwich beam of length L, the total weight is to be minimized, i.e.,

$$J = \int_0^L (2\rho_1 f + \rho_2 c)\, dx$$

In order to formulate the design as an optimal problem of Mayer, following the notation of Ref. 4, we let

$$u_1 = f = \text{face thickness}, \quad u_2 = c = \text{core thickness}$$

$$y_1 = w_b, \qquad y_2 = w_s$$

$$y_3 = \dot{w}_b = \text{slope of partial deflection due to bending}$$

$$\dot{y}_4 = (2\rho_1 u_1 + \rho_2 u_2) = \text{specific weight}$$

Then

$$J = y_4(L) \quad \text{with} \quad y_4(0) = 0 \tag{9}$$

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The state equations for a statically determinate sandwich beam are

$$\dot{y}_1 = y_3 \tag{10a}$$

$$\dot{y}_2 = Qu_2/G_c(u_1+u_2)^2 \tag{10b}$$

$$\dot{y}_3 = 2M/E_f\, u_1(u_1+u_2)^2 \tag{10c}$$

$$\dot{y}_4 = 2\rho_1 u_1 + \rho_2 u_2 \tag{10d}$$

The minimum weight design problem can be stated as that of determining u_1 and u_2 so that J is minimized while the state equations and associated boundary conditions and other requirements (constraints) are satisfied.

Inequality constraints of any or all of the types given below may be included in the problem definition.

a) Maximum Bending Stress

$$C_1 \equiv \left\{\left[\frac{M(x)}{u_1(x)[u_1(x)+u_2(x)]}\right]^2 - \sigma_{\max}{}^2\right\} \leq 0$$

b) Maximum Shearing Stress

$$C_2 \equiv \left\{\left[\frac{Q(x)}{[u_1(x)+u_2(x)]}\right]^2 - \tau_{\max}{}^2\right\} \leq 0$$

c) Minimum Face Thickness

$$C_3 \equiv a - u_1(x) \leq 0$$

d) Minimum Core Thickness

$$C_4 \equiv b - u_2(x) \leq 0$$

Using the techniques and terminology of optimal control, the design problem of statically determinate beams is reduced to the following multipoint boundary value problem. Vector-matrix notation is used.

Index of Performance (Scalar):

$$J = y_4(L) \tag{11}$$

Terminal Conditions (ψ is a 4 vector):

$$\psi[y(0), y(L), L] = 0 \tag{12a}$$

and

$$\psi_5[y(x_*), x_*] \equiv y_1(x_*) + y_2(x_*) - Y = 0 \tag{12b}$$

where x_* is some arbitrary point, e.g., $x_* = L$. In some problems the equality constraint of Eq. (12b) may be enforced at one end of the beam, in these cases, the equality constraint will be one of the four terminal conditions.

Constraint Equations (C is a q vector):

$$C[y(x), u(x), x] \leq 0 \tag{13}$$

Hamiltonian (Scalar):

$$H = \lambda^T(x) f[y(x), u(x), x] + \mu^T(x) C[y(x), u(x), x] \tag{14}$$

where

$$\begin{aligned} \mu_i(x) &= 0 \quad \text{if} \quad C_i[y(x), u(x), x] < 0 \\ \mu_i(x) &\neq 0 \quad \text{if} \quad C_i[y(x), u(x), x] = 0 \end{aligned} \tag{15}$$

State Equations (y and f are 4 vectors, u is a 2 vector):

$$dy/dx = f[y(x), u(x), x] \tag{16}$$

Adjoint Equations (λ is a 4 vector):

$$d\lambda/dx = -\lambda^T(\partial f/\partial y) - \mu^T(\partial C/\partial y) \tag{17}$$

Control Equations:

$$\lambda^T\, \partial f/\partial u + \mu^T\, \partial C/\partial u = 0 \tag{18}$$

Transversality Conditions:

$$dy_4(L) - H\, dx\Big|_0^L + \lambda^T dy\Big|_0^L = 0 \tag{19}$$

with

$$d\psi = 0 \tag{20}$$

Corner Conditions:

$$\lambda\Big|_{x_1(+)} = \lambda\Big|_{x_1(-)}, \qquad H\Big|_{x_1(+)} = H\Big|_{x_1(-)} \tag{21}$$

where x_1 is a corner point, i.e., a point of entry to or exit from a control boundary. If in Eq. (12b) $x_* \neq L$ the additional corner conditions

$$\lambda\Big|_{x_*(+)} = \lambda\Big|_{x_*(-)} + \nu, \qquad H\Big|_{x_*(+)} = H\Big|_{x_*(-)} \tag{22}$$

are utilized. ν is an unknown multiplier.

The equations listed previously comprise a set of necessary conditions for the minimization of the objective function J subject to all constraints. Examination of this system reveals that it consists of a set of eight ordinary differential equations in eight unknowns. A total of eight terminal conditions are specified, four from the terminal constraints of Eq. (12a), and four from the transversality condition of Eq. (19). The two control variables are determined from Eq. (18) if none of the constraints of Eq. (13) is active ($\mu_i = 0$). If some of these constraints are active, the control vector and values of μ are determined by solution of the equality condition of the active constraint and Eq. (18). If Eq. (12a) is not part of the terminal conditions of Eq. (12), an additional unknown constant ν is introduced by Eq. (22).

The optimization problem has thus been reduced to the solution of a two-point boundary value problem in the case of $x_* = L$ or a three-point boundary value problem if $x_* \neq L$. Problems of this type can be solved numerically by the Newton-Raphson Technique.[5]

Several example problems have been solved. The detailed formulation and results are given in the backup paper. Our experience indicated that the Newton-Raphson method converged rapidly and was relatively insensitive to choice of initial estimates. Computer run time for these examples was on the order of 3–5 sec of CDC 6500 time. Statically indeterminate problems can easily be treated by the introduction of auxiliary state variables which represent the magnitude of the reactions due to the indeterminancy.

References

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